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A note on Seiberg-Witten central charge.

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Abstract

The central charge for the Seiberg-Witten low-energy effective Action is computed using Noether supercharges. A reliable method to construct supersymmetric Noether currents is presented.

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The electromagnetic duality of Seiberg-Witten [1] relies heavily on the BPS mass-formula

$$M = |Z| \quad \text{where} \quad Z = (n_e a + n_m a_D) \tag{1}$$

is the topological central charge of the N=2 supersymmetry[1]. In Ref. [2] the classical Z was computed and it was conjectured that if the fields belong to the *small* representation of the N=2 supersymmetry they should saturate the BPS mass-formula, giving $M = |Z|$, even at the quantum level. But so far the only direct evidence of this is a BPS-type computation [3] of the minimum of the Hamiltonian, neglecting fermionic fields[2]. As stressed by Seiberg and Witten, another way to find the modification to the classical BPS formula is to compute the central charge Z from the low-energy U(1) effective Action[3] because only the *massless* (and neutral) degrees of freedom contribute to Z.

A way to compute the central charge Z is from the commutation relations of the supersymmetric Noether charges. The purpose of this paper is to present the result of such a computation for the low-energy U(1) part of the effective Action. The computation is by no means trivial but at the end the formula (1) is exactly what is obtained. Apart from the result itself, it is illuminating to see precisely how the centre remains *formally insensitive* to the quantum corrections.

As for any space-time symmetry, the Noether procedure for supersymmetry needs to be handled with care because of the variation $\delta\mathcal{L} = \partial_\mu V^\mu$ of the Lagrangian density and we present below a reliable way to write supercurrents taking this term carefully into account. We also show how to commute the charges, paying due attention to the way they are expressed in terms of fields and momenta.

In this letter we shall give only a brief account of the computation with special emphasis on the application of the Noether technique to Seiberg-Witten centre, leaving to later work a detailed discussion [6]. One of the most important results of the detailed discussion is that the general form of a supersymmetric Noether current is

[1]Duality means $\begin{pmatrix} a_D \\ a \end{pmatrix} \to D \begin{pmatrix} a_D \\ a \end{pmatrix}$ and $(n_m\ n_e) \to (n_m\ n_e)D^{-1}$ where $D \in SL(2, Z)$.

[2]The kind of result we are considering also seems to follow from a geometric analysis of the N=2 vector multiplet in Ref. [4], where, however, the authors' aim there is completely different, the fermionic contribution is not present and there is no mention of Noether charges.

[3]Despite the fact that the *massive* SU(2) gauge theory does not have the exact duality symmetry (due to the presence [5] of the scalar-vector-superfield coupling $\phi^\dagger e^V$), one would also like to calculate the Z for this theory, both to check that the U(1) argument given above is correct, and that the freedom to make a linear shift $\mathcal{F}(A) \to \mathcal{F}(A) + cA$ which appears in the U(1) theory is removed at the full SU(2) level. The SU(2) case will be treated in a later paper[6]

$$J^\mu = \Pi_i^\mu \delta\Phi^i \; - \; V^\mu \tag{2}$$

where the Φ_i's are the fields in the Lagrangian density $\mathcal{L}$ and $\Pi_i^\mu \equiv \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi^i}$. We shall call the $\Pi\delta\Phi$ part of the current the *rigid* part, as this is the only part that contributes for rigid internal symmetries. In the case of space-time symmetries we also have V^μ, where $\delta\mathcal{L} = \partial_\mu V^\mu$ and the variation of the Lagrangian density is taken *without* using the Euler-Lagrange equations for the fields. This situation will be familiar to the reader from the case of space-time translation symmetry where V_μ leads to the well-known term $-g_{\mu\nu}\mathcal{L}$ in the energy-momentum tensor[4]. For supersymmetric transformations the situation is more complicated simply because V_μ is more complicated and a major part of the task is to determine this quantity. Of course, V_μ is related to the second-last term in the superfield expansion, but knowing this does not appreciably simplify its computation. A second task is to write the full J^μ in terms of fields and their conjugate momenta[5]. When J^μ is written in this way the transformations, the Hamiltonian and the central charge can be obtained from the canonical commutation relations. It is the result of these calculations that we wish to present in this letter.

For the Seiberg-Witten centre the relevant formula is

$$\{Q_{1\alpha}\ ,\ Q_{2\beta}\} = 2i\epsilon_{\alpha\beta}\, Z \qquad Q_{L\alpha} = \int d^3x J^0_{L\alpha}(x) \quad L = 1, 2 \tag{3}$$

where $\epsilon_{12} = 1 = -\epsilon_{21}$ is understood, the Lie brackets $\{,\}$ stand for Poisson brackets and the charges $Q_{1\alpha}$ and $Q_{2\beta}$ are the generators of the two supersymmetries in $N = 2$. It is actually necessary to compute only $Q_{1\alpha}$, because $Q_{2\alpha}$ can be obtained from it by an R-transformation i.e. by letting $\psi \leftrightarrow -\lambda$ and $v_\mu \to -v_\mu$. The charges $Q_{L\alpha}$ come from the low-energy U(1) effective Lagrangian density which in component fields, up to four fermions and second derivatives of the fields, is given by

$$\begin{aligned}\mathcal{L} \;\; = \;\; & \mathrm{Im}\Big[- \mathcal{F}''(A)[\partial_\mu A^\dagger \partial^\mu A + \frac{1}{4} v_{\mu\nu} v^{\mu\nu} + \frac{i}{8} v_{\mu\nu} v^{*\mu\nu} + i\psi \not{\partial}\bar{\psi} + i\lambda \not{\partial}\bar{\lambda} - (F^\dagger F + \frac{1}{2} D^2)] \\ & + \mathcal{F}'''(A)[\frac{1}{\sqrt{2}} \lambda\sigma^{\mu\nu}\psi v_{\mu\nu} - \frac{1}{2}(F\psi^2 + F^\dagger\lambda^2) + \frac{i}{\sqrt{2}} D\psi\lambda] + \mathcal{F}''''(A)[\frac{1}{4}\psi^2\lambda^2]\Big]\end{aligned} \tag{4}$$

[4]In the case of supersymmetry transformations we have to enlarge the space-time to a *super*-space with grassmanian coordinates if we want to explicitly see those transformations as "space-time". Our approach is based on standard space-time and we do not have an analog of $V^\mu = -\mathcal{L}\delta x^\mu$ therefore we derived the current by comparing the variation *off-shell* with the variation *on-shell* [7].

[5]The difficulty of expressing Noether space-time currents, and therefore *super*-currents, in terms of fields and their conjugate momenta was pointed out to me by Steven Weinberg.

where A, ψ, F and v_μ, λ, D are the chiral and vector N=1 multiplets respectively, we have chosen the temporal gauge $v_o = 0$, fermions are in Weyl notation and $\mathcal{F}(A)$ is the holomorphic prepotential which classically reduces to $\frac{1}{2}\tau A^2$, with the complex coupling constant[6] $\tau = \tau_R + i\tau_I = \frac{\theta}{8\pi^2} + \frac{i}{g^2}$.

As we wish to compare the classical and effective central charges of the N=2 supersymmetry, we first re-obtain the classical result by starting from the classical version of (4), namely

$$\mathcal{L} = \mathrm{Im}\Big[-\tau[\partial_\mu A^\dagger \partial^\mu A + \frac{1}{4} v_{\mu\nu} v^{\mu\nu} + \frac{i}{8} v_{\mu\nu} v^{*\mu\nu} + i\psi \not{\partial} \bar{\psi} + i\lambda \not{\partial} \bar{\lambda} - (F^\dagger F + \frac{1}{2} D^2)]\Big] \tag{5}$$

By direct inspection we find for $Q_{1\alpha}$ supersymmetry

$$\Pi_i^\mu \delta_1 \Phi^i = \pi_A^\mu \delta_1 A + \Pi^{\mu\nu} \delta_1 v_\nu + \delta_1 \bar{\psi} \pi_{\bar{\psi}}^\mu \tag{6}$$

$$V_1^\mu = \pi_A^\mu \delta_1 A - \frac{i}{2} \tau^* \epsilon_1 \sigma_\nu \bar{\lambda} v^{*\mu\nu} \tag{7}$$

where we have written the conjugate momenta for each of the dynamical fields (in the temporal gauge for v_μ). Thus for the vector field $\Pi^{\mu\nu} = \frac{i}{2}(\tau \hat{v}^{\mu\nu} - \tau^* \hat{v}^{\dagger\mu\nu})$, where $\hat{v}^{\mu\nu}$ and $\hat{v}^{\dagger\mu\nu}$ are the self-dual and antiself-dual projections of $v^{\mu\nu}$ respectively, and for the fermions $\bar{\psi}$ and $\bar{\lambda}$ are regarded as the fields and $i\tau_I \bar{\sigma}^0 \psi$ and $i\tau_I \bar{\sigma}^0 \lambda$ as their conjugate momenta.

According to (2) the full supersymmetry current and charge then read

$$J_1^\mu = \Pi^{\mu\nu} \delta_1 v_\nu + \delta_1 \bar{\psi} \pi_{\bar{\psi}}^\mu + \frac{i}{2} \tau^* \epsilon_1 \sigma_\nu \bar{\lambda} v^{*\mu\nu} \tag{8}$$

and

$$Q_{1\alpha} = \int d^3x \Big(\Pi^i \delta_{1\alpha} v_i + \delta_{1\alpha} \bar{\psi} \pi_{\bar{\psi}} + \frac{i}{2} \tau^* (\sigma_\nu \bar{\lambda})_\alpha v^{*0i} \Big) \tag{9}$$

respectively. Note that in the total current the scalar terms $\pi_A \delta_1 A$ have canceled.

The charge $Q_{1\alpha}$ correctly generates the ϵ_1 supersymmetry transformations of the fields: for v_i and $\bar{\psi}$ this is obvious; for A we see that $\delta_1 \bar{\psi}$ contains π_A and the commutation gives the right factors; for λ and ψ one has to realize that $\delta_1 v_i$ contains $\bar{\lambda}$ and that we are looking for the transformations of the momenta $\pi_{\bar{\lambda}} = i\tau_I \bar{\sigma}^0 \lambda$ and $\pi_{\bar{\psi}} = i\tau_I \bar{\sigma}^0 \psi$. An

[6] Note that our τ is normalized to $\tau/4\pi$ of Seiberg-Witten. Note also that we keep $\theta \neq 0$ only as a computational tool even if the physics is not affected by it.

important feature to notice is that the Noether charges automatically produce the *on-shell* transformations [6].

By commuting the charge $Q_{1\alpha}$ with the charge $Q_{2\alpha}$ obtained by R-symmetry according to Eq. (3) we obtain, as might be expected, a total divergence, namely

$$\begin{aligned} \{Q_{1\alpha}\ ,\ Q_{2\beta}\} &= -\int d^3x \partial_i \Big[(2\sqrt{2}\Pi^i A^\dagger + \sqrt{2} v^{*0i} A_D^\dagger)\epsilon_{\alpha\beta} + \tau^* \epsilon^{0ijk}(\sigma_j \bar{\lambda})_\alpha (\sigma_k \bar{\psi})_\beta\Big] \\ &= -\epsilon_{\alpha\beta} 2\sqrt{2} \int d^2\vec{\Sigma} \cdot (\vec{\Pi} A^\dagger + \vec{B} A_D^\dagger) \end{aligned} \tag{10}$$

where $d^2\vec{\Sigma}$ is the measure on the sphere at infinity, $B^i = \frac{1}{2}\epsilon^{0ijk} v_{jk}$ and $A_D^\dagger = \tau^* A^\dagger$ is the classical analogue of the *dual* of the scalar field, which in Seiberg-Witten is $A_D^\dagger = \mathcal{F}^{\dagger\prime}(A^\dagger)$. We have made the usual assumption that $\bar{\psi}$ and $\bar{\lambda}$ fall off at least like $r^{-\frac{3}{2}}$ and have implemented the Gauss law as an identity.

For the effective Lagrangian (4) much more labor is needed to write the variation of $\mathcal{L}$ as a pure divergence[6]. At the end of the computation the *rigid* current and V^μ turn out to be

$$\Pi_i^\mu \delta_1 \Phi^i = \pi_A^\mu \delta_1 A + \Pi^{\mu\nu}\delta_1 v_\nu + \delta_1 \bar{\psi} \pi_{\bar{\psi}}^\mu \tag{11}$$

$$V_1^\mu = \pi_A^\mu \delta_1 A - \frac{i}{2}\mathcal{F}''^\dagger \epsilon_1 \sigma_\nu \bar{\lambda} v^{*\mu\nu} + \frac{1}{2\sqrt{2}}\mathcal{F}'''^\dagger \epsilon_1 \sigma^\mu \bar{\psi}\bar{\lambda}^2 \tag{12}$$

Here again we have written the conjugate momenta of the fields. We notice that the *rigid* current is formally the same as in the classical case and that V_1^μ differs only in the last term (containing $\bar{\psi}\bar{\lambda}^2$). Thus these currents correctly reduce to the ones in (6) and (7) when the classical limit is taken: $\mathcal{F}(A) \to \frac{1}{2}\tau A^2$. However, the formal resemblance masks the fact that the momenta and $\mathcal{F}''^\dagger$ are quite different when expressed as explicit functions of the fields and their derivatives. In particular the conjugate momentum of v_μ has a much more complicated expression involving all the fermions:
$\Pi^{\mu\nu} = \frac{i}{2}(\mathcal{F}''\hat{v}^{\mu\nu} - \mathcal{F}''^\dagger \hat{v}^{\dagger\mu\nu}) - \frac{i}{\sqrt{2}}(\mathcal{F}'''\lambda\sigma^{\mu\nu}\psi - \mathcal{F}'''^\dagger \bar{\lambda}\bar{\sigma}^{\mu\nu}\bar{\psi})$.
According to (2) the full current is then

$$J_1^\mu = \Pi^{\mu\nu}\delta_1 v_\nu + \delta_1 \bar{\psi}\pi_{\bar{\psi}}^\mu + \frac{i}{2}\mathcal{F}''^\dagger \epsilon_1 \sigma_\nu \bar{\lambda} v^{*\mu\nu} - \frac{1}{2\sqrt{2}}\mathcal{F}'''^\dagger \epsilon_1 \sigma^\mu \bar{\psi}\bar{\lambda}^2 \tag{13}$$

where once again the scalar contributions $\pi_A \delta_1 A$ have canceled. The charge is

$$Q_{1\alpha} = \int d^3x \Big(\Pi^i \delta_{1\alpha} v_i + \delta_{1\alpha}\bar{\psi}\pi_{\bar{\psi}} + \frac{i}{2}\mathcal{F}''^{\dagger}(\sigma_\nu\bar{\lambda})_\alpha v^{*0i} - \frac{1}{2\sqrt{2}}\mathcal{F}'''^{\dagger}(\sigma^0\bar{\psi})_\alpha\bar{\lambda}^2 \Big) \qquad (14)$$

Again this charge reproduces the right transformations by simply applying the same procedure outlined in the classical case and carefully handling the cubic fermion terms, which recombine to give the *on-shell* dummy fields [6].

According to Eq. (3) we commute the charge $Q_{1\alpha}$ with the charge $Q_{2\alpha}$ obtained by R-symmetry, and find that

$$\begin{aligned} \{Q_{1\alpha}\ ,\ Q_{2\beta}\} &= -\int d^3x \partial_i \Big[(2\sqrt{2}\Pi^i A^\dagger + \sqrt{2} v^{*0i} A_D^\dagger)\epsilon_{\alpha\beta} + \mathcal{F}^{\dagger''}\epsilon^{0ijk}(\sigma_j\bar{\lambda})_\alpha(\sigma_k\bar{\psi})_\beta \Big] \\ &= -\epsilon_{\alpha\beta} 2\sqrt{2} \int d^2\vec{\Sigma} \cdot (\vec{\Pi}A^\dagger + \vec{B}A_D^\dagger) \end{aligned} \qquad (15)$$

where again we have made the usual assumption that the fermion fields drop off at least like $r^{-\frac{3}{2}}$ and have implemented the Gauss law as an identity.

Note that the two expressions (10) and (15) have the same *form* and they only differ by replacing classical fields and momenta (and their dual) by their quantum counterparts. Of course this does not mean that the two centres are equal (as is expected to be the case for N=4 supersymmetry where the beta function is identically zero) but only that supersymmetry has protected the classical *form* and therefore the BPS mass-formula equally applies to the quantum case.

Finally one has to evaluate (10) and (15) explicitly in the limit $A^\dagger \to a^*$ and $A_D^\dagger \to a_D^*$ for $r \to \infty$ where a^* and a_D^* are c-numbers, and are non-zero in the spontaneously broken case. Up to an irrelevant numerical scale-factor [7] we obtain

$$Z = n_e a^* + n_m a_D^* \qquad (16)$$

in agreement with (1).

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[7] $Z \to (i\sqrt{2}\alpha)^{-1} Z$ and α depends on the conventions one uses to compute the integrals.

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